Schlessinger's criterion

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Chapter 1

Introduction

The goal of this project is to formalize Schlessinger's criterion in Lean following the original article [?]. Some parts are also based on the proof given in [?, Tag 06G7].

Chapter 2

Definitions

2.1 Base category

Let Λ be a complete Noetherian (commutative) local ring and k its residue field.

Definition 1. We define $\mathbf{C}_\Lambda$ to be the category of Artinian local Λ-algebras having residue field k, where morphisms are local algebra homomorphisms.

The condition of having the same residue field for a Λ-algebra A can be stated in a more general context as follows: the algebra map $\Lambda \to k_A$ is surjective (where k_A is the residue field of A). As such the conditions on Λ are omitted in Lean.

Furthermore, we do not need to assume the morphisms to be local, as we have the following result:

Lemma 2. *Let A and B be commutative local rings, and $f : A \to B$ a ring morphism. If A is Artinian, then f is local.*

Proof. In an Artinian local ring, the maximal ideal is the set of nilpotent elements. In particular if $x \in \mathfrak{m}_A$, then $f(x)$ is not a unit. □

The Artinian condition on elements of $\mathbf{C}_\Lambda$ could potentially be interpreted in two ways: either as Artinian rings or as Artinian Λ-modules. In this context, both are equivalent.

Lemma 3. *Let A be a local Λ-algebra with same residue field as Λ. Then* $\mathrm{length}_\Lambda(A) = \mathrm{length}_A(A)$.

Proof. We prove a more general result: if M is an A-module, then $\mathrm{length}_\Lambda(M) = \mathrm{length}_A(M)$. This result follows easily by induction on $\mathrm{length}_\Lambda(M)$ from the case $\mathrm{length}_\Lambda(M) = 1$, in which case we have $M \cong k$.

[**?**, Tag 06GG] □

Lemma 4. *The category $\mathbf{C}_\Lambda$ has pullbacks.*

Proof. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{C}_\Lambda$. Consider the set $P = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$. Since f and g are local, the subset $\mathfrak{m}_X \times \mathfrak{m}_Y$ is an ideal (which is maximal). We have $k_\Lambda \to k_P \to k_X$, so $k_P \cong k$. Furthermore, since X and Y are Artinian Λ-modules by theorem 3, P is an Artinian Λ-module as well, so it is an Artinian ring. □

We also note that k is a terminal object in $\mathbf{C}_\Lambda$, so the category also admits products (which are the same as pullbacks over the terminal object).

2.2 Small Extensions

Let $p : B \to A$ be a surjective morphism in $\mathbf{C}_\Lambda$.

Definition 5. p is a *small extension* if $\ker p$ is a non-zero principal ideal such that $\mathfrak{m}_B \cdot \ker p = 0$.

Lemma 6. *The following propositions are equivalent:*

1. *p is a small extension*
2. *$\ker p$ is a minimal non-zero ideal*
3. *$\mathrm{length}_\Lambda(B) = \mathrm{length}_\Lambda(A) + 1$*

Proof. Since p is surjective, we have $\mathrm{length}_\Lambda(B) = \mathrm{length}_\Lambda(A) + \mathrm{length}_\Lambda(\ker p)$, and we also have that $\mathrm{length}_B(\ker p) = 1$ if and only if $\ker p$ is a minimal non-zero ideal, so using theorem 3 we get (2) $\iff$ (3). We now show (1) $\iff$ (2).

Assume that p is a small extension, write $\ker p = (t)$, let $I < \ker p$ be an ideal and $x \in I$. There exists $y \in B$ such that $x = ty$. Since $I \neq (t)$, y is not a unit, so we have $y \in \mathfrak{m}_B$, and $x = 0$ by definition of a small extension. So $I = 0$ and $\ker p$ is minimal.

Assume that $\ker p$ is a minimal non-zero ideal. Let $t \in \ker p \setminus \{0\}$. By minimality, we have $\ker p = (t)$. Furthermore, since $\mathfrak{m}_B$ is nilpotent and $\ker p \neq 0$ we have $\mathfrak{m}_B \cdot \ker p \neq \ker p$ so $\mathfrak{m}_B \cdot \ker p = 0$ by minimality. So p is a small extension. □

Lemma 7. *If a property C on morphisms of $\mathbf{C}_\Lambda$ is satisfied by isomorphisms and stable by composition on the left by a small extension, then it holds for any surjection.*

Proof. By induction on $\mathrm{length}_\Lambda(B) \in \mathbb{N}$. If $\ker p = 0$, then p is an isomorphism. Otherwise, there exists a minimal non-zero ideal I of B because B is Artinian. By minimality, since $\ker p \neq 0$, we have $I \leq \ker p$, so p factorizes through the small extension $B \to B/I$. We conclude by applying the induction hypothesis to the map $B/I \to A$. □

Lemma 8. *If a property C on objects of $\mathbf{C}_\Lambda$ is satisfied by k, stable by isomorphisms and stable by small extensions, then it holds for all objects of $\mathbf{C}_\Lambda$.*

Proof. One way is to apply the previous lemma to the map $p : B \to k$ (i.e. with the property on morphisms $q : X \to Y$ given by if $Y = k$ then $C(X)$). In the code the proof is essentially copied and adapted from the previous lemma. □

2.3 Completed base category

For pro-representability, we consider a bigger category with objects that are projective limits of objects in the base category. However, in this context the term pro-representability is used in a more restrictive sense than usual, so that the ”completed” category is not the pro-category as one might expect.

Definition 9. A local ring with maximal ideal $\mathfrak{m}$ is *complete* if it is $\mathfrak{m}$-adically complete.

Definition 10. We define $\hat{\mathbf{C}}_\Lambda$ to be the category of complete Noetherian local Λ-algebras having residue field k, where morphisms are local algebra homomorphisms.

Note that a local ring is endowed with a natural topological ring structure, namely the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the maximal ideal. It is defined by setting the powers of $\mathfrak{m}$ as a neighborhood base of 0.

Lemma 11. *The category* $\hat{\mathbf{C}}_\Lambda$ *is a full subcategory of* $\mathbf{C}_\Lambda$*.*

Proof. An Artinian ring is Noetherian, and its $\mathfrak{m}$-adic topology is discrete, so it is trivially complete. □

Lemma 12. *A local Noetherian ring whose maximal ideal is nilpotent is Artinian.*

Proof. A Noetherian ring is Artinian if and only if it has Krull dimension 0, so it suffices to show that any prime ideal is maximal. Let I be a prime ideal. Since $\mathfrak{m}^n = 0 \leq I$ we have $\mathfrak{m} \leq I$, so $I = \mathfrak{m}$ is maximal. □

Lemma 13. *If* $A \in \hat{\mathbf{C}}_\Lambda$*, then for any* $n \in \mathbb{N}^*$ *we have* $A/\mathfrak{m}^n \in \mathbf{C}_\Lambda$ *where* $\mathfrak{m}$ *is the maximal ideal of* A*. In fact, we have* $A/J \in \mathbf{C}_\Lambda$ *for any ideal* J *such that* $\mathfrak{m}^n \leq J < A$*.*

Proof. $A/\mathfrak{m}^n$ is a local Noetherian ring, with nilpotent maximal ideal $\mathfrak{m}/\mathfrak{m}^n$ and residue field k, so theorem 12 applies. □

Remark. The fact that $A \in \hat{\mathbf{C}}_\Lambda$ is complete (and Noetherian) means that we have $A = \varprojlim A/\mathfrak{m}^n$. We don't really use this result here but it explains some of the underlying ideas.

Lemma 14. *Let* $R \in \hat{\mathbf{C}}_\Lambda$*,* $A \in \mathbf{C}_\Lambda$ *and* $u : R \to A$*. There exists* $n \in \mathbb{N}^*$ *such that* $\mathfrak{m}_R^n \leq \ker u$*, so* u *factorizes through* $u_n : R/\mathfrak{m}_R^n \to A$*.*

Proof. There exists n such that $\mathfrak{m}_A^n = 0$, so $\mathfrak{m}_R^n \leq u^{-1}(\mathfrak{m}_A^n) = \ker u$. □

2.4 Trivial Square-Zero Extensions

Definition 15. For V a k-vector space, we write $k[V]$ for the *trivial square-zero extension* of V over k, that is the module $k \oplus V$ with multiplication defined so that $V^2 = 0$.

Definition 16. We write $k[\varepsilon]$ for *dual numbers*, the special case where $V = k$.

Definition 17. We can a functor $\mathrm{Mod}_k^{fg} \to \mathbf{C}_\Lambda$ that commutes with finite products (where Mod_k^{fg} is the category of finite dimensional k-vector spaces).

Lemma 18. *If* F *is a functor* $\mathbf{C}_\Lambda \to$ *Sets such that*

$$F(k[V] \times k[W]) \tilde{\to} F(k[V]) \times F(k[W])$$

for all finite dimensional k*-vector spaces* V *and* W*, then* $F(k[V])$ *has a canonical* k*-vector space structure.*

Proof. This is a special case of the following lemma [**?**, Tag 06I6]: □

Lemma 19. *Let* R *be a (commutative) ring, and* $L : \mathrm{Mod}_R^{fg} \to$ *Sets be a functor that preserves finite products. Then for any* $M \in \mathrm{Mod}_R^{fg}$*,* $L(M)$ *has a canonical* R*-module structure.*

Proof. Let $M \in \mathrm{Mod}_R^{fg}$. Write $s : M \times M \to M$ for the addition map, $\lambda_r : M \to M$ for the scalar multiplication by $r \in R$, and $z : 0 \to M$. The module structure on $L(M)$ is defined using the images of these morphisms by L, together with the identification $L(M \times M) \cong L(M) \times L(M)$, and the fact that $L(0)$ is terminal in *Sets* so it is a singleton. Checking that this defines a module structure is rather straightforward, by writing suitable commutative diagrams in Mod_R^{fg} and pushing them with L. □

Lemma 20. *Let R be a (commutative) ring, L and $L' : \mathrm{Mod}_R^{fg} \to Sets$ be two functors that preserve finite products, and $\eta : L \to L'$ a natural transformation. Then for any $M \in \mathrm{Mod}_R^{fg}$, η_M is R-linear.*

Proof. This is a very simple check. $\square$

Lemma 21. *Let R be a (commutative) ring, L and $L' : \mathrm{Mod}_R^{fg} \to Sets$ be two functors that preserve finite products, and $\eta : L \to L'$ a natural transformation. Then η is an isomorphism if and only if η_R is an isomorphism.*

Proof. By induction on the dimension. $\square$

2.5 Artin Functors

Definition 22. For $R \in \hat{\mathbf{C}}_\Lambda$ and $A \in \mathbf{C}_\Lambda$, we define a functor $\mathbf{C}_\Lambda \to Sets$ by $h_R(A) = \mathrm{Hom}_{\hat{\mathbf{C}}_\Lambda}(R, A)$.

In what follows we will only consider functors F from $\mathbf{C}_\Lambda$ to *Sets* that preserve terminal objects, or equivalently such that $F(k)$ is a singleton.

Definition 23. We can extend such a functor to $\hat{\mathbf{C}}_\Lambda$ by setting for $R \in \hat{\mathbf{C}}_\Lambda$, $\hat{F}(R) = \varprojlim F(R/\mathfrak{m}^n)$. A map $u : R \to S$ induces maps $u_n : R/\mathfrak{m}_R^n \to S/\mathfrak{m}_S^n$ for all $n \in \mathbb{N}$, so we can also define $\hat{F}(u)$ by using the universal property. This defines a functor $\hat{F} : \hat{\mathbf{C}}_\Lambda \to Sets$.

Lemma 24. *There is a natural isomorphism $\hat{F}|_{\mathbf{C}_\Lambda} \cong F$.*

Proof. If A is in $\mathbf{C}_\Lambda$, there exists $N \in \mathbb{N}^*$ such that $\mathfrak{m}_A^N = 0$, so we have $F(A/\mathfrak{m}_A^n) \cong F(A)$ for all $n \geq N$. $\square$

Definition 25. A *pro-couple* is a pair (R, ξ) where $R \in \hat{\mathbf{C}}_\Lambda$ and $\xi \in \hat{F}(R)$.

Definition 26. A pro-couple (R, ξ) defines a natural transformation $\nu(\xi) : h_R \to F$ given by $u \mapsto \hat{F}(u)(\xi)$ up to the isomorphism given in lemma 24.

Lemma 27. *Let (R, ξ) be a pro-couple, $A, B \in \mathbf{C}_\Lambda$, $u : R \to A$ and $v : A \to B$. Then $\nu(\xi)(v \circ u) = F(v)(\nu(\xi)(u))$.*

Proof. This is essentially the naturality of $\hat{F}|_{\mathbf{C}_\Lambda} \cong F$. $\square$

Lemma 28. *Let (R, ξ) be a pro-couple, with $\xi = (\xi_q)_{q \in \mathbb{N}}$, $n \in \mathbb{N}^*$, and $p : R \to R/\mathfrak{m}_R^n$ the quotient map. Then $\nu(\xi)(p) = \xi_n$.*

Proof. In fact, we have that $\hat{F}(p)$ is equal to the projection $\hat{F}(R) \to F(R/\mathfrak{m}_R^n)$ composed with the natural isomorphism $F \cong \hat{F}|_{\mathbf{C}_\Lambda}$. This is a simple check. $\square$

In particular, since any map $R \to A$ with $A \in \mathbf{C}_\Lambda$ factorizes through some $u_n : R/\mathfrak{m}^n \to A$ (theorem 14), we have $\nu(\xi)(u) = F(u_n)(\xi_n)$.

Definition 29. A pro-couple (R, ξ) *pro-represents* F if the natural transformation $\nu(\xi) : h_R \to F$ it induces is an isomorphism. F is *pro-representable* if there exists a pro-couple which pro-represents F.

Definition 30. Let $F, G : \mathbf{C}_\Lambda \to Sets$ be two functors that preserve terminal objects. A natural transformation $\eta : F \to G$ is *smooth* if for any surjection $p : B \to A$ in $\mathbf{C}_\Lambda$, the morphism

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective.

Lemma 31. *To check that a natural transformation is smooth, it is enough to check the condition on morphisms p that are small extensions.*

Proof. Apply theorem 7. Check that the property is stable by composition (note that the pullback is in *Sets*). □

Lemma 32. *Let $F, G : \mathbf{C}_\Lambda \to Sets$ be two functors that preserve terminal objects. If $\eta : F \to G$ is smooth, then η_A is surjective for any $A \in \mathbf{C}_\Lambda$.*

Proof. Apply the condition to $p : A \to k$ (using that $F(k)$ and $G(k)$ are singletons). □

Definition 33. The *tangent space* of a functor F is $t_F = F(k[\varepsilon])$. In the case $F = h_R$, we simply write t_R.

Definition 34. A pro-couple (R, ξ) is a *hull* of F if the natural transformation $\nu(\xi) : h_R \to F$ it induces is smooth and $\nu(\xi)_{k[\varepsilon]} : t_R \to t_F$ is an isomorphism.

Note that if $\eta : F \to G$ is an isomorphism on $k[\varepsilon]$, then it is an isomorphism on $k[V]$ for any finite dimensional k-vector space (lemma 21).

2.6 Schlessinger's conditions

Let F be a functor from $\mathbf{C}_\Lambda$ to *Sets* such that $F(k)$ is a singleton. For $f : X \to Z$ and $G : Y \to Z$ morphisms in $\mathbf{C}_\Lambda$, we consider the comparison morphism:

$$p_{f,g} : F(X \times_Z Y) \to F(X) \times_{F(Z)} F(Y) \tag{2.1}$$

Definition 35. (H_1) : if g is a small extension then $p_{f,g}$ is surjective.

Definition 36. (H_2) : if $Z = k$ and $Y = k[\varepsilon]$ then $p_{f,g}$ is bijective.

Definition 37. (H_3) : assuming (H_2), $\dim_k(t_F) < \infty$.

Definition 38. (H_4) : if f is a small extension then $p_{f,f}$ is bijective.

These conditions are on the functor F, so for example "F satisfies (H_1)" means that for any morphisms f and g, if g is a small extension, then $p_{f,g}$ is surjective.

Lemma 39. *The definition of (H_3) makes sense because the condition (H_2) implies that F verifies the hypothesis of lemma 18, so that t_F admits a canonical k-vector space structure.*

Proof. □

In fact, we have the following result:

Lemma 40. *If F satisfies (H_2), we have $F(A \times k[V]) \cong F(A) \times F(k[V])$ for any $A \in \mathbf{C}_\Lambda$ and any $V \in \mathrm{Mod}_k^{fg}$.*

Proof. By induction on the dimension of V. □

We also have the following generalization from (H_1):

Lemma 41. *If F satisfies (H_1), then $p_{f,g}$ is surjective for any surjection g.*

Proof. A surjection g is either an isomorphism or it can be factored as a composition of small extensions. The result is clear for isomorphisms and we can show that if g_1 and g_2 are two suitable morphisms such that p_{f,g_1} and p_{f,g_2} are surjective then $p_{f,g_2 \circ g_1}$ is surjective as well. □

We will also need the following fact:

Lemma 42. *The functor h_R satisfies conditions (H_1), (H_2), (H_3) and (H_4).*

Proof. In fact in this case (H_1), (H_2) and (H_4) hold trivially since $p_{f,g}$ is an isomorphism for any morphisms f and g, because h_R is a hom-functor, and pullbacks in $\mathbf{C}_\Lambda$ are pullbacks in $\hat{\mathbf{C}}_\Lambda$. For (H_3), it suffices to show the following linear isomorphism $t_R^* \cong \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda R)$. □

Chapter 3

Schlessinger's theorem

Let F be a functor from $\mathbf{C}_\Lambda$ to *Sets* such that $F(k)$ is a singleton.

Theorem 43. *If F satisfies (H_1), (H_2) and (H_3), then F admits a hull.*

Theorem 44. *If F admits a hull, then F satisfies (H_1), (H_2) and (H_3).*

Theorem 45. *If F satisfies (H_1), (H_2), (H_3) and (H_4), then F is pro-representable.*

Theorem 46. *If F is pro-representable, then F satisfies (H_1), (H_2), (H_3) and (H_4).*

Proof. h_R satisfies all four conditions as noted in theorem 42, so this is trivial (using theorem 20 for (H_3)). □

The main difficulty is the backward direction of (1).

Chapter 4

Lemmas

We now give some additional lemmas needed for the proof.

4.1 Essential morphisms

Definition 47. A surjective morphism $p : B \to A$ in $\mathbf{C}_\Lambda$ is *essential* if for any morphism $q : C \to B$ such that $q \circ p$ is surjective, q is surjective.

Lemma 48. *Let $p : B \to A$ be essential and $q : C \to B$ be a morphism in $\hat{\mathbf{C}}_\Lambda$ such that $q \circ p$ is surjective. Then q is surjective.*

Proof. This is trivial considering that q factorizes through $q_n : C/\mathfrak{m}_C^n \to B$ for some n which is a morphism in $\mathbf{C}_\Lambda$ (theorem 14). $\square$

Lemma 49. *If $p : B \to A$ is a small extension, then p is not essential if and only if p admits a section, that is there exists a morphism $s : A \to B$ such that $p \circ s = \mathbb{1}_A$.*

Proof. If p has a section s, then p is not essential since s is not surjective as we have $\text{length}_\Lambda(B) = \text{length}_\Lambda(A) + 1$.

Now, suppose that p is not essential, and let $q : C \to B$ be such that $q \circ p$ is surjective but q is not. Let C' be the range of q. We have $\text{length}_\Lambda(C') < \text{length}_\Lambda(B)$ since q is not surjective so $\text{length}_\Lambda(C') \leq \text{length}_\Lambda(A)$, and $\text{length}_\Lambda(C') \geq \text{length}_\Lambda(A)$ because $p|_{C'}$ is surjective (because $q \circ p$ is). Thus $p|_{C'}$ is bijective which gives the section that we wanted. $\square$

4.2 Action of a small extension

Let $p : B \to A$ be a small extension in $\mathbf{C}_\Lambda$, and write $I = \ker p$.

Lemma 50. *There is an isomorphism $B \times k[I] \cong B \times_A B$ whose first component is the identity.*

Proof. It is given by $(x, r + y) \mapsto (x, x + y)$. The inverse map is $(x, y) \mapsto (x, x_0 + y - x)$ where x_0 is the k residue of x. It is easy to check that this defines an isomorphism. $\square$

Using this isomorphism, we can get the following result as in [**?**, Remark 2.15]:

Lemma 51. *Let $F : \mathbf{C}_\Lambda \to Sets$ such that $F(k)$ is a singleton and F satisfies (H_2). Using the isomorphism from lemma 50 together with the comparison maps (eq. (2.1)), we get a map*

$$F(B) \times F(k[I]) \to F(B) \times_{F(A)} F(B)$$

This defines an action of $F(k[I])$ on $F(B)$, with orbits contained in the fibers of p.

Proof. This is simply a formal check. In practice we only need the fact that 0 acts trivially, so there is less to check. □

Lemma 52. *The condition (H_1) implies that the map in theorem 51 is surjective, i.e. the action is transitive on fibers of p.*

Proof. □

Lemma 53. *The condition (H_4) implies that the map in theorem 51 is bijective, i.e. the fibers of p are principal homogeneous spaces under $F(k[I])$.*

Proof. □

Lemma 54. *The action defined in theorem 51 is functorial: if G is another functor with the necessary conditions, and $\eta : F \to G$ is a natural transformation, then the following diagram commutes:*

$$\begin{array}{ccc} G(B) \times G(k[I]) & \longrightarrow & G(B) \times_{G(A)} G(B) \\ \uparrow & & \uparrow \\ F(B) \times F(k[I]) & \longrightarrow & F(B) \times_{F(A)} F(B) \end{array}$$

Proof. □

4.3 Topology

We need the following lemma [**?**, Tag 06SE]:

Lemma 55. *Let R be an object of $\hat{\mathbf{C}}_\Lambda$, and let $(J_n)_{n\in\mathbb{N}}$ be a decreasing sequence of ideals of R such that $\mathfrak{m}_R^n \leq J_n$. Set $J = \bigcap_{n\in\mathbb{N}} J_n$. Then the sequence $(J_n/J)_{n\in\mathbb{N}}$ defines the $\mathfrak{m}_{R/J}$-adic topology on R/J.*

Proof. We need to show that for any $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that $J_N \leq J + \mathfrak{m}_R^n$. For every $k \in \mathbb{N}$, $R/\mathfrak{m}_R^k$ is Artinian, so the decreasing sequence $(J_i + \mathfrak{m}_R^k)_{i\in\mathbb{N}}$ stabilizes. We write N_k for some index such that $J_{N_k} + \mathfrak{m}_R^k$ is the minimum of the sequence. Note that we have $J_{N_k} + \mathfrak{m}_R^k \leq J_k + \mathfrak{m}_R^k$ so since we also have $\mathfrak{m}_R^k \leq J_k$ we get that $J_{N_k} \leq J_k$. We claim that $J_{N_n} \leq J + \mathfrak{m}_R^n$. Let $x \in J_{N_n}$. We define a Cauchy sequence (x_i) in R with $x_i \in J_{N_i}$ for $i \geq n$ by induction, by first setting $x_i = x$ if $i \leq n$. If $i > n$, then there exists $x_{i+1} \in J_{N_{i+1}}$ such that $x_i - x_{i+1} \in \mathfrak{m}_R^i$: indeed if $N_{i+1} \leq N_i$ we can take $x_{i+1} = x_i$, and otherwise we have $J_{N_{i+1}} + \mathfrak{m}_R^i = J_{N_i} + \mathfrak{m}_R^i$ (note that have $x_i \in J_{N_i}$). Since R is complete, this sequence admits a limit y. We have $y \in J_{N_i} + \mathfrak{m}_R^i$ for all i so $y \in J$ and $y - x = y - x_n \in \mathfrak{m}_R^n$, so $x \in J + \mathfrak{m}_R^n$. □

In particular, this means that if $J \neq R$, we have $\hat{F}(R/J) \cong \varprojlim F(R/J_n)$.

Chapter 5

Proof

In this section we prove the main theorem (chapter 3).

5.1 Existence of a hull

The main proof is the following, showing the existence of a hull given (H_1), (H_2) and (H_3).

Proof.

Suppose that F satisfies (H_1), (H_2) and (H_3). Let $r = \dim_k(t_F)$, $S = \Lambda[[X_1, \dots, X_r]]$ and $\mathfrak{n}$ the maximal ideal of S.

We first construct R as a projective limit of quotients of S. More specifically we define a sequence $(J_q, \xi_q)_{q\geq 2}$ by induction where (J_q) is a decreasing sequence of ideals of S such that $R_q = S/J_q \in \mathbf{C}_\Lambda$ and $\xi_q \in F(R_q)$ is such that $(\xi_q) \in \varprojlim F(R_q)$.

First we set $J_2 = \mathfrak{n}^2 + \mathfrak{m}_\Lambda S$ and $R_2 = S/J_2$.

There exists $\xi_2 \in R_2$ inducing an isomorphism $\mathrm{Hom}(R_2, k[\varepsilon]) \cong t_F$. To see this, first note that $R_2 \cong k[k^r] \cong k[\varepsilon]^r$. By applying (H_2), this yields $F(R_2) \cong t_F^r$, so choosing a basis $(e_1, \dots, e_r)$ of t_F gives an element $\xi_2 \in F(R_2)$. We then check that $u \mapsto F(u)(\xi_2)$ is an isomorphism. In fact, this map is linear and sends the morphism $R_2 \to k[\varepsilon]$ corresponding to the i-th projection $k^r \to k$ to e_i.

For the induction step, suppose that we have an ideal J_q such that $R_q = S/J_q \in \mathbf{C}_\Lambda$ and $\xi_q \in F(R_q)$. Consider the set $\mathcal{S}$ of ideals J of S such that $\mathfrak{n}J_q \leq J \leq J_q$ and ξ_q lifts to $F(S/J)$. We will set J_{q+1} to be the minimum of $\mathcal{S}$, with ξ_{q+1} some lift of ξ_q, after proving that it exists. First note that $\mathcal{S}$ is not empty as it contains J_q, and since $S/\mathfrak{n}J_q$ is Artinian, we know that $\mathcal{S}$ admits a minimal element, so we only need to show that $\mathcal{S}$ is stable by pairwise intersection to get a minimum. Let J and K be two elements of $\mathcal{S}$. The elements of $\mathcal{S}$ are in correspondence with k-vector subspaces of $J/\mathfrak{n}J_q$, so we can extend J into some ideal $J' \leq J_q$ such that $J' \cap K = J \cap K$ and $J' + K = J_q$, and we have $J' \in \mathcal{S}$ because $J \leq J'$. We now use the isomorphism

$$S/J' \times_{S/J_q} S/K \cong S/(J \cap K)$$

together with theorem 41 to conclude that ξ_q lifts to $S/(J \cap K)$, i.e. that $J \cap K \in \mathcal{S}$, which concludes the induction step.

We set $J = \bigcap_{q\geq 2} J_q$ and $R = S/J$. By definition of the J_q's, we have $\mathfrak{n}^q \leq J_q$, so theorem 55 applies and $\hat{F}(R) = \varprojlim F(S/J_q)$, so we can set $\xi = \varprojlim \xi_q \in \hat{F}(R)$.

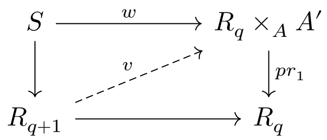
We have $\mathrm{Hom}(R, k[\varepsilon]) \cong \mathrm{Hom}(R_2, k[\varepsilon])$ because any map $S \to k[\varepsilon]$ factorizes through S/J_2, and since ξ projects to ξ_2 we have $t_R \cong t_F$ by definition of ξ_2.

It remains to show that $\nu(\xi) : h_R \to F$ is smooth.

Using theorem 31, let $p : A' \to A$ be a small extension, $u : R \to A$, $\eta' \in F(A')$ and $\eta \in F(A)$ such that $\nu(\xi)(u) = F(p)(\eta') = \eta$. We need to find $u' : R \to A'$ such that $\nu(\xi)(u') = \eta'$ and $p \circ u' = u$.

First we reduce this to finding $u'' : R \to A'$ such that $p \circ u'' = u$. Indeed, if we have such a u'', then $\nu(\xi)(u'')$ and η' both lie over η, so by theorem 52, there exists $\sigma \in F(k[I])$ sending $\nu(\xi)(u'')$ to η', where $I = \ker p$. By using theorem 54 and the fact that $h_R(k[I]) \cong F(k[I])$ (using theorem 21), we can also apply σ to u'' giving us the $u' : R \to A'$ that we wanted.

Since u factorizes through $R/\mathfrak{m}_R^n \to A$ for some n, and the topology is induced by the J_q's (theorem 55), u factorizes through $R_q \to A$ for some q. It suffices to find v completing the following diagram:



where w is defined using the universal property of $S = \Lambda[[X_1, \ldots, X_r]]$ to make the diagram commute. More specifically, to define w we use that pr_1 is surjective so that we can first define w on the X_i's, and since all maps are local the images of the X_i's are in the maximal ideal so we can extend the map to S. In this context we can easily show that pr_1 is a small extension, so we can apply theorem 49. If pr_1 admits a section then we are done. Otherwise, pr_1 is essential, and $pr_1 \circ w$ is the quotient map $S \to R_q$, so w is surjective by theorem 48. By (H_1), ξ_q lifts to $R_q \times_A A'$, so $\ker w \in \mathcal{S}$ and by minimality we have $J_{q+1} \leq \ker w$. Thus w factors through R_{q+1} giving the v that we wanted.

□

5.2 Necessity of the conditions

We now prove that the conditions (H_1), (H_2) and (H_3) are necessary for the existence of a hull.

Proof. Suppose that F admits a hull (R, ξ).

First since $t_R \cong t_F$ (linearly by theorem 20), and h_R satisfies (H_3), F satisfies (H_3).

To check (H_1), let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathbf{C}_\Lambda$ with g surjective, and $x \in F(X)$ and $y \in F(Y)$ both lying over $z \in F(Z)$. First, since $h_R \to F$ is smooth, by theorem 32 there exists $u_x : R \to X$ such that $\nu(\xi)(u_x) = x$. Then by smoothness applied to g, there exists $u_y : R \to Y$ such that $\nu(\xi)(u_y) = y$ and $g \circ u_y = f \circ u_x$. Then $\nu(\xi)(u_x \times u_y) \in F(X \times_Z Y)$ projects to x and y, so $p_{f,g}$ is surjective.

To check (H_2), we now only need to check injectivity, since we have already shown (H_1). Let $A \in \mathbf{C}_\Lambda$ and ζ_1 and ζ_2 in $F(A \times k[\varepsilon])$ having the same projections $a \in F(A)$ and $e \in k[\varepsilon]$. We proceed similarly as in the proof of (H_1). First there is $u' : R \to A$ such that $\nu(\xi)(u') = a$. Then by smoothness applied to the projection $pr_1 : A \times k[\varepsilon] \to A$, we get maps $u_i : R \to A \times k[\varepsilon]$ for $i = 1, 2$ such that $\nu(\xi)(u_i) = \zeta_i$ and $pr_1 \circ u_i = u'$. Since $\nu(\xi)(pr_2 \circ u_i) = e$ in both cases, and $\nu(\xi)_{k[\varepsilon]}$ is an isomorphism, we have $u_1 = u_2$ so $\zeta_1 = \zeta_2$.

□

5.3 Pro-representability

We finally prove that the conditions (H_1), (H_2), (H_3) and (H_4) are sufficient for pro-representability.

Proof. Suppose that F satisfies (H_1), (H_2), (H_3) and (H_4). By the first point of the theorem, we know that F admits a hull (R, ξ). Furthermore, since $\nu(\xi)$ is smooth, we already know that $\nu(\xi)_A$ is surjective for all $A \in \mathbf{C}_\Lambda$ (theorem 32). We will show by induction on $\text{length}_\Lambda(A)$ that it is injective, using **??**. If $A = k$, then this is true by definition of a hull. Otherwise, let $p : B \to A$ be a small extension, with $I = \ker p$, and suppose that $\nu(\xi)_A$ is an isomorphism. By applying theorem 53 together with theorem 54, it is easy check that $\nu(\xi)_B$ is injective. Namely we take u_1 and u_2 in $h_R(B)$ such that $\nu(\xi)(u_1) = \nu(\xi)(u_2)$, and we need to show that the action of $h_R(k[I])$ by 0 sends u_1 to u_2, which is clear from the commutative diagram in theorem 54. □

Chapter 6

Remaining `sorry`'s

Here is the list of all the remaining Lean statements using `sorry` at the time of writing, with a mathematical description and some ideas for the proofs.

theorem 21

Lemma 56. *Let R be a Noetherian ring. Then $R[[X]]$ is Noetherian.*

This is necessary for section 5.1 especially to show that some quotients of $S = \Lambda[[X_1, \dots, X_r]]$ are in $\mathbf{C}_\Lambda$. This is an important result that is currently missing from Mathlib. I have heard that someone was working on this, so I shouldn't need to prove this.

Lemma 57. *Let R be a local ring, $\mathfrak{m}$ its maximal ideal, $r, k \in \mathbb{N}$, $\mathfrak{n}$ the maximal ideal of $R[[X_1, \cdots, X_r]]$ and $x = (x_n)_{n \in \mathbb{N}^r} \in R[[X_1, \cdots, X_r]]$. Then $x \in \mathfrak{n}^k \iff \forall n \in \mathbb{N}^r, x_n \in \mathfrak{m}^{k-\deg(n)}$.*

Proof. The forward direction is easy. The converse should probably be shown by induction on r (and k). □

This lemma is very important to deal with $S = \Lambda[[X_1, \dots, X_r]]$, for example to show that S is complete when Λ is and to show that $R_2 \cong k[k^r]$ in section 5.1.

Lemma 58. *The functor h_R (see theorem 22) preserves pullbacks.*

See theorem 42.

Proof. It suffices to show that the inclusion functor preserves pullbacks. □

Lemma 59. *The functor h_R satisfies (H_3).*

See theorem 42.

Proof. I have shown that $h_R(k[\varepsilon]) = Hom(R, k[\varepsilon]) \cong Der_\Lambda(R, k[\varepsilon]) \cong (\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda R))^*$. The main missing part is the linearity of the first bijection. The fact that $\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\Lambda R)$ is finite dimensional is also missing for now. □

Lemma 60. *The map $Hom(k^r, k) \to Hom(k[k^r], k[\varepsilon])$ is k-linear ($r \in \mathbb{N}$). (k here is the residue field of Λ)*

The vector space structure on $Hom(k[k^r], k[\varepsilon])$ here is the one induced by the coyoneda functor $Hom(k[k^r], -)$ (theorem 18). This is somewhat similar to the previous point, where the abstract vector space structure is the main issue. This is used in section 5.1 to show the existence of ξ_2.

Lemma 61. *Let R be a Λ-algebra, and J, K ideals of R. Then,*

$$R/J \times_{R/(J+K)} R/K \cong R/(J \cap K)$$

This is the easiest `sorry`.

Lemma 62. *In the proof of the theorem, we have $r \in \mathbb{N}$, $S = \Lambda[[X_1, \cdots, X_r]]$, and $J \neq S$ an ideal of S. S/J is complete (as a local ring).*

This is needed for section 5.1. It is essentially a topological result that is in Mathlib (as), however it is stated here in a way that doesn't mention topology, and in fact the topology has not been defined at all in Lean yet.

theorem 44 Only (H_3) has been shown for now. The proof is described in section 5.2.